Exercise 1.1.10

Test for convergence

$$\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2=\frac{1}{4}+\frac{9}{64}+\frac{25}{256}+\cdots.$$

Solution

Start by rewriting the infinite series.

$$\begin{aligned}\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2&=\sum_{n=1}^{\infty}\left[\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)(2n)}{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}\right]^2\\&=\sum_{n=1}^{\infty}\frac{[(2n)!]^2}{[2\cdot 4\cdot 6\cdots(2n)]^4}\\&=\sum_{n=1}^{\infty}\frac{[(2n)!]^2}{[2^n(1\cdot 2\cdot 3\cdots n)]^4}\\&=\sum_{n=1}^{\infty}\frac{[(2n)!]^2}{2^{4n}(n!)^4}\end{aligned}$$

Since there are factorials and exponents involving n, use the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{\{[2(n+1)]!\}^2}{2^{4(n+1)}[(n+1)!]^4}}{\frac{[(2n)!]^2}{2^{4n}(n!)^4}}&=\lim_{n\to\infty}\frac{\{[2(n+1)]!\}^2}{2^{4(n+1)}[(n+1)!]^4}\times\frac{2^{4n}(n!)^4}{[(2n)!]^2}\\&=\lim_{n\to\infty}\frac{[(2n+2)!]^2}{2^{4n}2^4[(n+1)!]^4}\times\frac{2^{4n}(n!)^4}{[(2n)!]^2}\\&=\lim_{n\to\infty}\frac{(2n+2)^2(2n+1)^2}{2^4(n+1)^4}\\&=\frac{1}{16}\lim_{n\to\infty}\frac{n^2\left(2+\frac{2}{n}\right)^2n^2\left(2+\frac{1}{n}\right)^2}{n^4\left(1+\frac{1}{n}\right)^4}\\&=\frac{1}{16}\left[\frac{(2+0)^2(2+0)^2}{(1+0)^4}\right]\\&=1\end{aligned}$$

Since the limit is 1, the ratio test is inconclusive.

A more sensitive test, namely the Gauss test, is necessary.

$$\frac{a_n}{a_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2} \tag{1}$$

If $B(n)$ is bounded for large enough n, then $\sum_n a_n$ converges when $h > 1$ and diverges when $h \leq 1$. The goal, then, is to use long division to rewrite the ratio of a_n/a_{n+1}.

$$\begin{aligned}
\frac{a_n}{a_{n+1}} = \frac{\frac{[(2n)!]^2}{2^{4n}(n!)^4}}{\frac{\{[2(n+1)]!\}^2}{2^{4(n+1)}[(n+1)!]^4}} &= \frac{[(2n)!]^2}{2^{4n}(n!)^4} \times \frac{2^{4(n+1)}[(n+1)!]^4}{\{[2(n+1)]!\}^2} \\
&= \frac{[(2n)!]^2}{2^{4n}(n!)^4} \times \frac{2^{4n}2^4[(n+1)!]^4}{[(2n+2)!]^2} \\
&= \frac{2^4(n+1)^4}{(2n+2)^2(2n+1)^2} \\
&= \frac{16(n^4+4n^3+6n^2+4n+1)}{(4n^2+8n+4)(4n^2+4n+1)} \\
&= \frac{16n^4+64n^3+96n^2+64n+16}{16n^4+48n^3+52n^2+24n+4}
\end{aligned}$$

$$16n^4 + 48n^3 + 52n^2 + 24n + 4 \;\big)\overline{\;16n^4 + 64n^3 + 96n^2 + 64n + 16\;}$$

Multiply $16n^4$ by 1 to get $16n^4$. Then subtract 1 times the divisor from the dividend.

$$\begin{array}{r}
1 \\
16n^4 + 48n^3 + 52n^2 + 24n + 4 \;\big)\overline{\;16n^4 + 64n^3 + 96n^2 + 64n + 16\;} \\
-(16n^4 + 48n^3 + 52n^2 + 24n + 4)
\end{array}$$

Do the subtraction.

$$\begin{array}{r}
1 \\
16n^4 + 48n^3 + 52n^2 + 24n + 4 \;\big)\overline{\;16n^4 + 64n^3 + 96n^2 + 64n + 16\;} \\
-(16n^4 + 48n^3 + 52n^2 + 24n + 4) \\
\hline
16n^3 + 44n^2 + 40n + 12
\end{array}$$

Multiply $16n^4$ by $1/n$ to get $16n^3$. Then subtract $1/n$ times the divisor from the dividend.

$$
\begin{array}{r}
1+\frac{1}{n} \\
16n^4+48n^3+52n^2+24n+4 \;\big)\; 16n^4+64n^3+96n^2+64n+16 \\
-\left(16n^4+48n^3+52n^2+24n+4\right) \\
\hline
16n^3+44n^2+40n+12 \\
-\left(16n^3+48n^2+52n+24+\frac{4}{n}\right)
\end{array}
$$

Do the subtraction.

$$
\begin{array}{r}
1+\frac{1}{n} \\
16n^4+48n^3+52n^2+24n+4 \;\big)\; 16n^4+64n^3+96n^2+64n+16 \\
-\left(16n^4+48n^3+52n^2+24n+4\right) \\
\hline
16n^3+44n^2+40n+12 \\
-\left(16n^3+48n^2+52n+24+\frac{4}{n}\right) \\
\hline
-4n^2-12n-12-\frac{4}{n}
\end{array}
$$

Multiply $16n^4$ by $-1/(4n^2)$ to get $-4n^2$. Then subtract $-1/(4n^2)$ times the divisor from the dividend.

$$
\begin{array}{r|l}
 & 1 + \frac{1}{n} - \frac{1}{4n^2} \\
16n^4 + 48n^3 + 52n^2 + 24n + 4 & 16n^4 + 64n^3 + 96n^2 + 64n + 16 \\
 & -\left(16n^4 + 48n^3 + 52n^2 + 24n + 4\right) \\
\hline
 & 16n^3 + 44n^2 + 40n + 12 \\
 & -\left(16n^3 + 48n^2 + 52n + 24 + \frac{4}{n}\right) \\
\hline
 & -4n^2 - 12n - 12 - \frac{4}{n} \\
 & -\left(-4n^2 - 12n - 13 - \frac{6}{n} - \frac{1}{n^2}\right)
\end{array}
$$

Do the subtraction.

$$
\begin{array}{r|l}
 & 1 + \frac{1}{n} - \frac{1}{4n^2} \\
16n^4 + 48n^3 + 52n^2 + 24n + 4 & 16n^4 + 64n^3 + 96n^2 + 64n + 16 \\
 & -\left(16n^4 + 48n^3 + 52n^2 + 24n + 4\right) \\
\hline
 & 16n^3 + 44n^2 + 40n + 12 \\
 & -\left(16n^3 + 48n^2 + 52n + 24 + \frac{4}{n}\right) \\
\hline
 & -4n^2 - 12n - 12 - \frac{4}{n} \\
 & -\left(-4n^2 - 12n - 13 - \frac{6}{n} - \frac{1}{n^2}\right) \\
\hline
 & 1 + \frac{2}{n} + \frac{1}{n^2}
\end{array}
$$

Multiply $16n^4$ by $1/(16n^4)$ to get 1. Then subtract $1/(16n^4)$ times the divisor from the dividend.

$$
\begin{array}{r|l}
 & 1+\frac{1}{n}-\frac{1}{4n^2}+\frac{1}{16n^4} \\
\hline
16n^4+48n^3+52n^2+24n+4 & 16n^4+64n^3+96n^2+64n+16 \\
 & -\left(16n^4+48n^3+52n^2+24n+4\right) \\
\cline{2-2}
 & 16n^3+44n^2+40n+12 \\
 & -\left(16n^3+48n^2+52n+24+\frac{4}{n}\right) \\
\cline{2-2}
 & -4n^2-12n-12-\frac{4}{n} \\
 & -\left(-4n^2-12n-13-\frac{6}{n}-\frac{1}{n^2}\right) \\
\cline{2-2}
 & 1+\frac{2}{n}+\frac{1}{n^2} \\
 & -\left(1+\frac{3}{n}+\frac{13}{4n^2}+\frac{3}{2n^3}+\frac{1}{4n^4}\right)
\end{array}
$$

Do the subtraction.

$$
\begin{array}{r|l}
 & 1 + \frac{1}{n} - \frac{1}{4n^2} + \frac{1}{16n^4} - \dots \\
\hline
16n^4 + 48n^3 + 52n^2 + 24n + 4 & 16n^4 + 64n^3 + 96n^2 + 64n + 16 \\
 & -\left(16n^4 + 48n^3 + 52n^2 + 24n + 4\right) \\
\hline
 & 16n^3 + 44n^2 + 40n + 12 \\
 & -\left(16n^3 + 48n^2 + 52n + 24 + \frac{4}{n}\right) \\
\hline
 & -4n^2 - 12n - 12 - \frac{4}{n} \\
 & -\left(-4n^2 - 12n - 13 - \frac{6}{n} - \frac{1}{n^2}\right) \\
\hline
 & 1 + \frac{2}{n} + \frac{1}{n^2} \\
 & -\left(1 + \frac{3}{n} + \frac{13}{4n^2} + \frac{3}{2n^3} + \frac{1}{4n^4}\right) \\
\hline
 & -\frac{1}{n} - \frac{9}{4n^2} - \frac{3}{2n^3} - \frac{1}{4n^4}
\end{array}
$$

Based on the new dividend, there would follow a term with $1/n^5$ and so on.

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} - \frac{1}{4n^2} + \frac{1}{16n^4} - \cdots$$

$$= 1 + \frac{1}{n} + \frac{-\frac{1}{4} + \frac{1}{16n^2} - \cdots}{n^2}$$

Comparing this with equation (1), $-\frac{1}{4} + \frac{1}{16n^2} - \cdots$ is bounded as n becomes large, and $h = 1$. Therefore,

$$\sum_{n=1}^{\infty} \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right]^2 = \frac{1}{4} + \frac{9}{64} + \frac{25}{256} + \cdots .$$

diverges by the Gauss test.